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Berry phase in generalized chiral QED_2

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Abstract. We consider the generalized chiral QED_2 on S^1 with a U(1) gauge field coupled with different charges to both chiral components of a fermionic field. Using the adiabatic approximation we calculate the Berry phase and the corresponding U(1) connection and curvature for the vacuum and many particle Fock states. We show that the nonvanishing vacuum Berry phase is associated with a projective representation of the local gauge symmetry group and contributes to the effective action of the model.

1. Introduction

Gauge models with anomaly are interesting from different points of view. First, there is a problem of consistent quantization for these models. Due to anomaly some constraints change their nature after quantization: instead of being first-class constraints, they turn into second-class ones. A consistent canonical quantization scheme clearly should take into account such a change [1–4].

Next is a problem of the relativistic invariance. It is known that in the physical sector where the local gauge invariance holds the relativistic invariance is broken for some anomalous models, namely the chiral Schwinger model (CSM) and chiral QED_2 [5–7]. For both models the Poincare algebra commutation relations breaking term can be constructed explicitly [7].

In the present paper we address ourselves to another aspect of anomalous models: the Berry phase and its connection to anomaly. A common topological nature of the Berry phase, or more generally quantum holonomy, and gauge anomalies was noted in [8, 3]. The former was shown to be crucial in the Hamiltonian interpretation of anomalies.

We consider a general version of the CSM with a U(1) gauge field coupled with different charges to both chiral components of a fermionic field. The nonanomalous Schwinger model (SM) where these charges are equal is a special case of the generalized CSM. This will allow us to see any distinction between the models with and without anomaly.

We suppose that space is a circle of length L, $-\frac{L}{2} \leqslant x < \frac{L}{2}$, so the spacetime manifold is a cylinder $S^1 \otimes R^1$. We work in the temporal gauge $A_0 = 0$ and use the system of units where $c = 1$. Only matter fields are quantized, while A_1 is handled as a classical background field. Our aim is to calculate the Berry phase and the corresponding U(1) connection and curvature for the fermionic Fock vacuum as well as for many particle states constructed over the vacuum and to show explicitly a connection between the nonvanishing vacuum Berry phase and anomaly.

Our paper is organized as follows. In section 2, we apply first and second quantization to the matter fields and obtain the second quantized fermionic Hamiltonian. We define

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the Fock vacuum and construct many particle Fock states over the vacuum. We use a particle-hole interpretation for these states.

In section 3, we first derive a general formula for the Berry phase and then calculate it for the vacuum and many particle states. We show that for all Fock states the Berry phase vanishes in the case of models without anomaly. We discuss a connection between the nonvanishing vacuum Berry phase, anomaly and effective action of the model.

Our conclusions are in section 4.

2. Quantization of matter fields

The Lagrangian density of the generalized CSM is

$$\mathcal{L} = -\tfrac{1}{4}\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu} + \bar{\psi}\mathrm{i}\hbar\gamma^{\mu}\partial_{\mu}\psi + e_{+}\bar{\psi}_{+}\gamma^{\mu}\psi_{+}A_{\mu} + e_{-}\bar{\psi}_{-}\gamma^{\mu}\psi_{-}A_{\mu} \tag{1}$$

where

$$\mathrm{F}_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \qquad (\mu,\nu) = \overline{0,1} \qquad \gamma^0 = \sigma_1$$
$$\gamma^1 = -\mathrm{i}\sigma_2 \qquad \gamma^0\gamma^1 = \gamma^5 = \sigma_3$$

only σ_i $(i = \overline{1,3})$ are Pauli matrices. The field ψ is 2-component Dirac spinor, $\bar{\psi} = \psi^{\dagger}\gamma^0$ and $\psi_{\pm} = \frac{1}{2}(1 \pm \gamma^5)\psi$.

In the temporal gauge $A_0 = 0$, the Hamiltonian density is

$$\mathcal{H} = \tfrac{1}{2}\mathrm{E}^2 + \mathcal{H}_{+} + \mathcal{H}_{-} \tag{2}$$

with E momentum canonically conjugate to A_1, and

$$\mathcal{H}_{\pm} \equiv \psi_{\pm}^{\dagger}d_{\pm}\psi_{\pm} = \mp\psi_{\pm}^{\dagger}(\mathrm{i}\hbar\partial_1 + e_{\pm}A_1)\psi_{\pm}.$$

On the circle boundary conditions for the fields must be specified. We impose the periodic ones

$$A_1\left(-\frac{\mathrm{L}}{2}\right) = A_1\left(\frac{\mathrm{L}}{2}\right)$$
$$\psi_{\pm}\left(-\frac{\mathrm{L}}{2}\right) = \psi_{\pm}\left(\frac{\mathrm{L}}{2}\right). \tag{3}$$

The Lagrangian and Hamiltonian densities are invariant under local time-independent gauge transformations

$$A_1 \to A_1 + \partial_1\lambda$$
$$\psi_{\pm} \to \exp\left\{\frac{\mathrm{i}}{\hbar}e_{\pm}\lambda\right\}\psi_{\pm}$$

λ being a gauge function.

For arbitrary e_{+}, e_{-}, the gauge transformations do not respect the boundary conditions (3). The gauge transformations compatible with the boundary conditions must be either of the form

$$\lambda\left(\frac{\mathrm{L}}{2}\right) = \lambda\left(-\frac{\mathrm{L}}{2}\right) + \hbar\frac{2\pi}{e_{+}}n \qquad n \in \mathcal{Z}$$

with $e_{+} \neq 0$ and

$$\frac{e_{-}}{e_{+}} = N \qquad N \in \mathcal{Z} \tag{4}$$

or of the form

$$\lambda\left(\frac{\mathrm{L}}{2}\right)=\lambda\left(-\frac{\mathrm{L}}{2}\right)+\hbar\frac{2\pi}{e_-}n \qquad n\in\mathcal{Z}$$

with $e_-\neq 0$ and

$$\frac{e_+}{e_-}=\bar{N} \qquad \bar{N}\in\mathcal{Z}. \tag{5}$$

Equations (4) and (5) imply a quantization condition for the charges. Without loss of generality, we choose (4). For $N=1$, $e_-=e_+$ and we have the standard SM. For $N=0$, we get the model in which only the positive chirality component of the Dirac field is coupled to the gauge field.

We see that the gauge transformations under consideration are divided into topological classes characterized by the integer n. If $\lambda(\frac{\mathrm{L}}{2})=\lambda(-\frac{\mathrm{L}}{2})$, then the gauge transformation is topologically trivial and belongs to the $n=0$ class. If $n\neq 0$ it is nontrivial and has winding number n.

The eigenfunctions and the eigenvalues of the first quantized fermionic Hamiltonians are

$$d_{\pm}\langle x|n;\pm\rangle=\pm\varepsilon_{n,\pm}\langle x|n;\pm\rangle$$

where

$$\langle x|n;\pm\rangle=\frac{1}{\sqrt{\mathrm{L}}}\exp\left\{\frac{\mathrm{i}}{\hbar}e_{\pm}\int_{-\mathrm{L}/2}^{x}\mathrm{d}z\,A_1(z)+\frac{\mathrm{i}}{\hbar}\varepsilon_{n,\pm}x\right\}$$

$$\varepsilon_{n,\pm}=\frac{2\pi}{\mathrm{L}}\left(n\hbar-\frac{e_{\pm}b\mathrm{L}}{2\pi}\right).$$

We see that the spectrum of the eigenvalues depends on the zero mode of the gauge field:

$$b\equiv\frac{1}{\mathrm{L}}\int_{-\mathrm{L}/2}^{\mathrm{L}/2}\mathrm{d}x\,A_1(x,t).$$

For $\frac{e_+b\mathrm{L}}{2\pi\hbar}=$ integer, the spectrum contains the zero energy level. As b increases from 0 to $\hbar\frac{2\pi}{e_+\mathrm{L}}$, the energies of $\varepsilon_{n,+}$ decrease by $\hbar\frac{2\pi}{\mathrm{L}}$, while the energies of $(-\varepsilon_{n,-})$ increase by $\hbar\frac{2\pi}{\mathrm{L}}\mathrm{N}$. Some of energy levels change sign. However, the spectrum at the configurations $b=0$ and $b=\hbar\frac{2\pi}{e_+\mathrm{L}}$ is the same, namely, the integers, as it must be since these gauge-field configurations are gauge-equivalent. In what follows, we will use separately the integer and fractional parts of $\frac{e_{\pm}b\mathrm{L}}{2\pi\hbar}$, denoting them as $[\frac{e_{\pm}b\mathrm{L}}{2\pi\hbar}]$ and $\{\frac{e_{\pm}b\mathrm{L}}{2\pi\hbar}\}$ correspondingly.

Now we introduce the second quantized right-handed and left-handed Dirac fields. For the moment, we will assume that $d_{\pm}$ do not have zero eigenvalues. At time $t=0$, in terms of the eigenfunctions of the first quantized fermionic Hamiltonians the second quantized (ζ-function regulated) fields have the expansion [9]

$$\begin{aligned}\psi_+^s(x)&=\sum_{n\in\mathcal{Z}}a_n\langle x|n;+\rangle|\lambda\varepsilon_{n,+}|^{-s/2}\\ \psi_-^s(x)&=\sum_{n\in\mathcal{Z}}b_n\langle x|n;-\rangle|\lambda\varepsilon_{n,-}|^{-s/2}.\end{aligned}\tag{6}$$

Here λ is an arbitrary constant with dimension of length which is necessary to make $\lambda\varepsilon_{n,\pm}$ dimensionless, while $a_n, a_n^{\dagger}$ and $b_n, b_n^{\dagger}$ are correspondingly right-handed and left-handed fermionic annihilation and creation operators which fulfil the commutation relations

$$[a_n,a_m^{\dagger}]_+=[b_n,b_n^{\dagger}]_+=\delta_{m,n}.$$

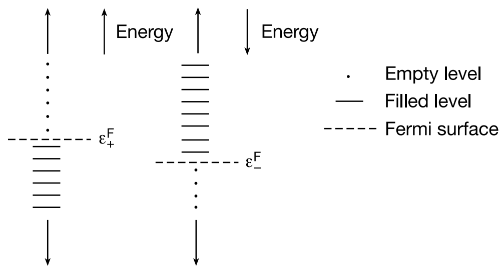

Figure 1. Schematic representation of the vacuum state: (*a*) positive chirality sector, (*b*) negative chirality sector.

For $\psi_\pm^s(x)$, the equal time anticommutators are

$$[\psi_\pm^s(x), \psi_\pm^{\dagger s}(y)]_+ = \zeta_\pm(s, x, y) \tag{7}$$

with all other anticommutators vanishing, where

$$\zeta_\pm(s, x, y) \equiv \sum_{n \in \mathcal{Z}} \langle x|n; \pm\rangle\langle n; \pm|y\rangle|\lambda\varepsilon_{n,\pm}|^{-s}$$

s is large and positive. In the limit, when the regulator is removed, i.e. $s = 0$, $\zeta_\pm(s = 0, x, y) = \delta(x - y)$ and equation (7) takes the standard form.

The vacuum state of the second quantized fermionic Hamiltonian

$$|\mathrm{vac}; A\rangle = |\mathrm{vac}; A; +\rangle \otimes |\mathrm{vac}; A; -\rangle$$

is defined such that all negative energy levels are filled and the others are empty (see figure 1):

$$\begin{aligned} a_n|\mathrm{vac}; A; +\rangle &= 0 \qquad \text{for } n > \left[\frac{e_+ b\mathrm{L}}{2\pi\hbar}\right] \\ a_n^\dagger|\mathrm{vac}; A; +\rangle &= 0 \qquad \text{for } n \leqslant \left[\frac{e_+ b\mathrm{L}}{2\pi\hbar}\right] \end{aligned} \tag{8}$$

and

$$\begin{aligned} b_n|\mathrm{vac}; A; -\rangle &= 0 \qquad \text{for } n \leqslant \left[\frac{e_- b\mathrm{L}}{2\pi\hbar}\right] \\ b_n^\dagger|\mathrm{vac}; A; -\rangle &= 0 \qquad \text{for } n > \left[\frac{e_- b\mathrm{L}}{2\pi\hbar}\right]. \end{aligned} \tag{9}$$

In other words, in the positive chirality vacuum all the levels with energy lower than $\varepsilon_{[\frac{e_+ b\mathrm{L}}{2\pi\hbar}]+1,+}$ and in the negative chirality one all the levels with energy lower than $(-\varepsilon_{[\frac{e_- b\mathrm{L}}{2\pi\hbar}],-})$ are filled:

$$|\mathrm{vac}; A; +\rangle = \prod_{n=-\infty}^{[\frac{e_+ b\mathrm{L}}{2\pi\hbar}]} a_m^\dagger|0; +\rangle$$

$$|\mathrm{vac}; A; -\rangle = \prod_{n=[\frac{e_- b\mathrm{L}}{2\pi\hbar}]+1}^{+\infty} b_n^\dagger|0; -\rangle$$

where $|0\rangle = |0, +\rangle \otimes |0, -\rangle$ is the state of 'nothing' with all the energy levels empty.

The Fermi surfaces which are defined to lie halfway between the highest filled and lowest empty levels are

$$\varepsilon_{\pm}^{\mathrm{F}} = \pm\hbar\frac{2\pi}{\mathrm{L}}\left(\frac{1}{2} - \left\{\frac{e_{\pm}b\mathrm{L}}{2\pi\hbar}\right\}\right).$$

For $e_+ = e_-$, $\varepsilon_+^{\mathrm{F}} = -\varepsilon_-^{\mathrm{F}}$.

Next we define the fermionic parts of the second-quantized Hamiltonian as

$$\hat{\mathrm{H}}_{\pm}^{s} = \int_{-\mathrm{L}/2}^{\mathrm{L}/2} \mathrm{d}x\,\hat{\mathcal{H}}_{\pm}^{s}(x) = \tfrac{1}{2}\int_{-\mathrm{L}/2}^{\mathrm{L}/2} \mathrm{d}x\,(\psi_{\pm}^{\dagger s} d_{\pm}\psi_{\pm}^{s} - \psi_{\pm}^{s} d_{\pm}^{\star}\psi_{\pm}^{\dagger s}).$$

Substituting (6) into this expression, we obtain

$$\hat{\mathrm{H}}_{\pm} = \hat{\mathrm{H}}_{0,\pm} \mp e_{\pm}b : \rho_{\pm}(0) : + \frac{1}{\hbar}\frac{\mathrm{L}}{4\pi}(\varepsilon_{\pm}^{\mathrm{F}})^2 \tag{10}$$

where double dots indicate normal ordering with respect to $|\mathrm{vac}, A\rangle$,

$$\hat{\mathrm{H}}_{0,+} = \hbar\frac{2\pi}{\mathrm{L}}\lim_{s\to 0}\left\{\sum_{k>[\frac{e_+b\mathrm{L}}{2\pi\hbar}]} k a_k^{\dagger}a_k|\lambda\varepsilon_{k,+}|^{-s} - \sum_{k\leqslant[\frac{e_+b\mathrm{L}}{2\pi\hbar}]} k a_k a_k^{\dagger}|\lambda\varepsilon_{k,+}|^{-s}\right\}$$

$$\hat{\mathrm{H}}_{0,-} = \hbar\frac{2\pi}{\mathrm{L}}\lim_{s\to 0}\left\{\sum_{k>[\frac{e_-b\mathrm{L}}{2\pi\hbar}]} k b_k b_k^{\dagger}|\lambda\varepsilon_{k,-}|^{-s} - \sum_{k\leqslant[\frac{e_-b\mathrm{L}}{2\pi\hbar}]} k b_k^{\dagger}b_k|\lambda\varepsilon_{k,-}|^{-s}\right\}$$

are free fermionic Hamiltonians, and

$$: \rho_{+}(0) := \lim_{s\to 0}\left\{\sum_{k>[\frac{e_+b\mathrm{L}}{2\pi\hbar}]} a_k^{\dagger}a_k|\lambda\varepsilon_{k,+}|^{-s} - \sum_{k\leqslant[\frac{e_+b\mathrm{L}}{2\pi\hbar}]} a_k a_k^{\dagger}|\lambda\varepsilon_{k,+}|^{-s}\right\}$$

$$: \rho_{-}(0) := \lim_{s\to 0}\left\{\sum_{k\leqslant[\frac{e_-b\mathrm{L}}{2\pi\hbar}]} b_k^{\dagger}b_k|\lambda\varepsilon_{k,-}|^{-s} - \sum_{k>[\frac{e_-b\mathrm{L}}{2\pi\hbar}]} b_k b_k^{\dagger}|\lambda\varepsilon_{k,-}|^{-s}\right\}$$

are charge operators for the positive and negative chirality fermion fields respectively. The fermion momentum operators constructed analogously are

$$\hat{\mathrm{P}}_{\pm} = \hat{\mathrm{H}}_{0,\pm}.$$

The operators $:\hat{\mathrm{H}}_{\pm}:$, $:\rho_{\pm}(0):$ and $\hat{\mathrm{P}}_{\pm}$ are well defined when acting on finitely excited states which have only a finite number of excitations relative to the Fock vacuum.

For the vacuum state,

$$:\hat{\mathrm{H}}_{\pm}: |\mathrm{vac}; A; \pm\rangle =: \rho_{\pm}(0): |\mathrm{vac}; A; \pm\rangle = 0.$$

Due to the normal ordering, the energy of the vacuum which is at the same time the ground state of the fermionic Hamiltonians turns out to be equal to zero (we neglect an infinite energy of the filled levels below the Fermi surfaces $\varepsilon_{\pm}^{\mathrm{F}}$). The vacuum state can be considered also as a state of the zero charge.

Any other state of the same charge will have some of the levels above $\varepsilon_+^{\mathrm{F}}$ $(\varepsilon_-^{\mathrm{F}})$ occupied and some levels below $\varepsilon_+^{\mathrm{F}}$ $(\varepsilon_-^{\mathrm{F}})$ unoccupied. It is convenient to use the vacuum state $|\mathrm{vac}; A\rangle$ as a reference, describing the removal of a particle of positive (negative) chirality from one of the levels below $\varepsilon_+^{\mathrm{F}}$ $(\varepsilon_-^{\mathrm{F}})$ as the creation of a 'hole' [10, 11]. Particles in the levels above $\varepsilon_+^{\mathrm{F}}$ $(\varepsilon_-^{\mathrm{F}})$ are still called particles. If a particle of positive (negative) chirality is excited from the level m below the Fermi surface to the level n above the Fermi surface, then we say that a hole of positive chirality with energy $(-\varepsilon_{m,+})$ and momentum $(-\hbar\frac{2\pi}{\mathrm{L}}m)$ (or of negative chirality with energy $\varepsilon_{m,-}$ and momentum $\hbar\frac{2\pi}{\mathrm{L}}m$) has been created as well as the positive chirality particle with energy $\varepsilon_{n,+}$ and momentum $\hbar\frac{2\pi}{\mathrm{L}}n$ (or the negative chirality one with energy $(-\varepsilon_{n,-})$ and momentum $(-\hbar\frac{2\pi}{\mathrm{L}}n)$). The operators $a_k(k \leqslant [\frac{e_+b\mathrm{L}}{2\pi\hbar}])$ and $b_k(k > [\frac{e_-b\mathrm{L}}{2\pi\hbar}])$ behave like creation operators for the positive and negative chirality holes correspondingly.

In the charge operator a hole counts as -1, so that, for example, any state with one particle and one hole as well as the vacuum state has vanishing charge.

The number of particles and holes of positive and negative chirality outside the vacuum state is given by the operators

$$\mathrm{N}_+ = \lim_{s\to 0}\left\{\sum_{k>[\frac{e_+b\mathrm{L}}{2\pi\hbar}]} a_k^\dagger a_k + \sum_{k\leqslant[\frac{e_+b\mathrm{L}}{2\pi\hbar}]} a_k a_k^\dagger\right\}|\lambda\varepsilon_{k,+}|^{-s}$$

$$\mathrm{N}_- = \lim_{s\to 0}\left\{\sum_{k\leqslant[\frac{e_-b\mathrm{L}}{2\pi\hbar}]} b_k^\dagger b_k + \sum_{k>[\frac{e_-b\mathrm{L}}{2\pi\hbar}]} b_k b_k^\dagger\right\}|\lambda\varepsilon_{k,-}|^{-s}$$

which count both particle and hole as $+1$.

Excited states are constructed by operating creation operators on the vacuum. We start with 1-particle states. Let us define the states $|m; A; \pm\rangle$ as follows

$$|m; A; +\rangle \equiv \begin{cases} a_m^\dagger|\mathrm{vac}; A; +\rangle & \text{for } m > \left[\dfrac{e_+b\mathrm{L}}{2\pi\hbar}\right] \\ a_m|\mathrm{vac}; A; +\rangle & \text{for } m \leqslant \left[\dfrac{e_+b\mathrm{L}}{2\pi\hbar}\right] \end{cases}$$

and

$$|m; A; -\rangle \equiv \begin{cases} b_m^\dagger|\mathrm{vac}; A; -\rangle & \text{for } m \leqslant \left[\dfrac{e_-b\mathrm{L}}{2\pi\hbar}\right] \\ b_m|\mathrm{vac}; A; -\rangle & \text{for } m > \left[\dfrac{e_-b\mathrm{L}}{2\pi\hbar}\right]. \end{cases}$$

The states $|m; A; \pm\rangle$ are orthonormalized,

$$\langle m; A; \pm|n, A; \pm\rangle = \delta_{mn}$$

and fulfil the completeness relation

$$\sum_{m\in\mathcal{Z}} |m; A; \pm\rangle\langle m; A; \pm| = 1.$$

It is easily checked that

$$:\hat{\mathrm{H}}_\pm: |m; A; \pm\rangle = \varepsilon_{m,\pm}|m; A; \pm\rangle$$
$$\hat{\mathrm{P}}_\pm|m; A; \pm\rangle = \hbar\frac{2\pi}{\mathrm{L}}m|m; A; \pm\rangle$$
$$:\rho_\pm(0): |m; A; \pm\rangle = \pm|m; A; \pm\rangle \qquad \text{for } m > \left[\frac{e_\pm b\mathrm{L}}{2\pi\hbar}\right]$$

and

$$:\hat{\mathrm{H}}_\pm: |m; A; \pm\rangle = -\varepsilon_{m,\pm}|m; A; \pm\rangle$$
$$\hat{\mathrm{P}}_\pm|m; A; \pm\rangle = -\hbar\frac{2\pi}{\mathrm{L}}m|m; A; \pm\rangle$$
$$:\rho_\pm(0): |m; A; \pm\rangle = \mp|m; A; \pm\rangle \qquad \text{for } m \leqslant \left[\frac{e_\pm b\mathrm{L}}{2\pi\hbar}\right].$$

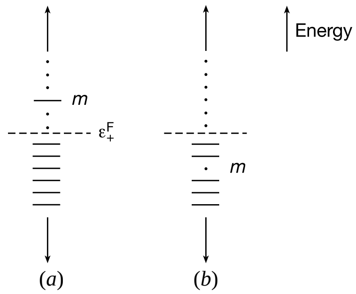

Figure 2. Schematic representation of 1-particle states: (a) a state with one particle, (b) a state with one hole. Only positive chirality sector is shown.

We see that $|m; A; +\rangle$ is a state with one particle of positive chirality with energy $\varepsilon_{m,+}$ and momentum $\hbar\frac{2\pi}{L}m$ for $m > [\frac{e_+bL}{2\pi\hbar}]$ or a state with one hole of the same chirality with energy $(-\varepsilon_{m,+})$ and momentum $(-\hbar\frac{2\pi}{L}m)$ for $m \leqslant [\frac{e_+bL}{2\pi\hbar}]$. The negative chirality state $|m; A; -\rangle$ is a state with one particle with energy $(-\varepsilon_{m,-})$ and momentum $(-\hbar\frac{2\pi}{L}m)$ for $m \leqslant [\frac{e_-bL}{2\pi\hbar}]$ or a state with one hole with energy $\varepsilon_{m,-}$ and momentum $\hbar\frac{2\pi}{L}m$ for $m > [\frac{e_-bL}{2\pi\hbar}]$. In any case,

$$N_\pm|m; A; \pm\rangle = |m; A; \pm\rangle$$

that is why $|m; A; \pm\rangle$ are called 1-particle states (see figure 2).

By applying n creation operators to the vacuum states $|\mathrm{vac}; A; \pm\rangle$ we can also get n-particle states

$$|m_1; m_2; \ldots; m_n; A; \pm\rangle \qquad (m_1 < m_2 < \cdots < m_n)$$

which are orthonormalized:

$$\langle m_1; m_2; \ldots; m_n; A; \pm|\overline{m}_1; \overline{m}_2; \ldots; \overline{m}_n; A; \pm\rangle = \delta_{m_1\overline{m}_1}\delta_{m_2\overline{m}_2}\ldots\delta_{m_n\overline{m}_n}.$$

The completeness relation is written in the following form

$$\frac{1}{n!}\sum_{m_1\in\mathcal{Z}}\cdots\sum_{m_n\in\mathcal{Z}}|m_1; m_2; \ldots; m_n; A; \pm\rangle\langle m_1; m_2; \ldots; m_n; A; \pm| = 1. \qquad (11)$$

Here the range of m_i ($i = \overline{1, n}$) is not restricted by the condition $(m_1 < m_2 < \cdots < m_n)$, duplication of states being taken care of by the $1/n!$ and the normalization. The '1' on the right-hand side of equation (11) means the unit operator on the space of n-particle states.

The case $n = 0$ corresponds to the zero-particle states. They form a one-dimensional space, all of whose elements are proportional to the vacuum state.

The multiparticle Hilbert space is a direct sum of an infinite sequence of the n-particle Hilbert spaces. The states of different numbers of particles are defined to be orthogonal to each other.

The completeness relation in the multiparticle Hilbert space has the form

$$\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{m_1,m_2,\ldots m_n\in\mathcal{Z}}|m_1; m_2; \ldots; m_n; A; \pm\rangle\langle m_1; m_2; \ldots; m_n; A; \pm| = 1 \qquad (12)$$

where '1' on the right-hand side means the unit operator on the whole multiparticle space.

For n-particle states,

$$:\hat{\mathrm{H}}_{\pm}:|m_1;m_2;\ldots;m_n;A;\pm\rangle=\sum_{k=1}^{n}\varepsilon_{m_k,\pm}\mathrm{sign}(\varepsilon_{m_k,\pm})|m_1;m_2;\ldots;m_n;A;\pm\rangle$$

and

$$:\rho_{\pm}(0):|m_1;m_2;\ldots;m_n;A;\pm\rangle=\pm\sum_{k=1}^{n}\mathrm{sign}(\varepsilon_{m_k,\pm})|m_1;m_2;\ldots;m_n;A;\pm\rangle.$$

3. Calculation of Berry phases

In the adiabatic approach [12–14], the dynamical variables are divided into two sets, one which we call fast variables and the other which we call slow variables. In our case, we treat the fermions as fast variables and the gauge fields as slow variables.

Let $\mathcal{A}^1$ be a manifold of all static gauge field configurations $A_1(x)$. On $\mathcal{A}^1$ a time-dependent gauge field $A_1(x,t)$ corresponds to a path and a periodic gauge field to a closed loop.

We consider the fermionic part of the second-quantized Hamiltonian $:\hat{\mathrm{H}}_{\mathrm{F}}:=:\hat{\mathrm{H}}_{+}:+:\hat{\mathrm{H}}_{-}:$ which depends on t through the background gauge field A_1 and so changes very slowly with time. We consider next the periodic gauge field $A_1(x,t)(0\leqslant t<T)$. After a time, T, the periodic field, $A_1(x,t)$, returns to its original value: $A_1(x,0)=A_1(x,T)$, so that $:\hat{\mathrm{H}}_{\pm}:(0)=:\hat{\mathrm{H}}_{\pm}:(T)$.

At each instant t we define eigenstates for $:\hat{\mathrm{H}}_{\pm}:(t)$ by

$$:\hat{\mathrm{H}}_{\pm}:(t)|\mathrm{F},A(t);\pm\rangle=\varepsilon_{\mathrm{F},\pm}(t)|\mathrm{F},A(t);\pm\rangle.$$

The state $|\mathrm{F}=0,A(t);\pm\rangle\equiv|\mathrm{vac};A(t);\pm\rangle$ is a ground state of $:\hat{\mathrm{H}}_{\pm}:(t)$,

$$:\hat{\mathrm{H}}_{\pm}:(t)|\mathrm{vac};A(t);\pm\rangle=0.$$

The Fock states $|\mathrm{F},A(t)\rangle=|\mathrm{F},A(t);+\rangle\otimes|\mathrm{F},A(t);-\rangle$ depend on t only through their implicit dependence on A_1. They are assumed to be orthonormalized,

$$\langle\mathrm{F}',A(t)|\mathrm{F},A(t)\rangle=\delta_{\mathrm{F},\mathrm{F}'}$$

and nondegenerate.

The time evolution of the wavefunction of our system (fermions in a background gauge field) is clearly governed by the Schrodinger equation:

$$\mathrm{i}\hbar\frac{\partial\psi(t)}{\partial t}=:\hat{\mathrm{H}}_{\mathrm{F}}:(t)\psi(t).$$

For each t, this wavefunction can be expanded in terms of the 'instantaneous' eigenstates $|\mathrm{F},A(t)\rangle$.

Let us choose $\psi_{\mathrm{F}}(0)=|\mathrm{F},A(0)\rangle$, i.e. the system is initially described by the eigenstate $|\mathrm{F},A(0)\rangle$. According to the adiabatic approximation, if at $t=0$ our system starts in a stationary state $|\mathrm{F},A(0)\rangle$ of $:\hat{\mathrm{H}}_{\mathrm{F}}:(0)$, then it will remain, at any other instant of time t, in the corresponding eigenstate $|\mathrm{F},A(t)\rangle$ of the instantaneous Hamiltonian $:\hat{\mathrm{H}}_{\mathrm{F}}:(t)$. In other words, in the adiabatic approximation transitions to other eigenstates are neglected.

Thus, at some time t later our system will be described up to a phase by the same Fock state $|\mathrm{F},A(t)\rangle$:

$$\psi_{\mathrm{F}}(t)=\mathrm{C}_{\mathrm{F}}(t)|\mathrm{F},A(t)\rangle$$

where $\mathrm{C}_{\mathrm{F}}(t)$ is yet an undetermined phase.

To find the phase, we insert $\psi_{\rm F}(t)$ into the Schrodinger equation:

$$\hbar \dot{\rm C}_{\rm F}(t) = -{\rm i}{\rm C}_{\rm F}(t)(\varepsilon_{{\rm F},+}(t) + \varepsilon_{{\rm F},-}(t)) - \hbar {\rm C}_{\rm F}(t) \left\langle {\rm F}, A(t) \left| \frac{\partial}{\partial t} \right| {\rm F}, A(t) \right\rangle .$$

Solving this equation, we obtain

$${\rm C}_{\rm F}(t) = \exp \left\{ -\frac{\rm i}{\hbar} \int_0^t {\rm d}t' \, (\varepsilon_{{\rm F},+}(t') + \varepsilon_{{\rm F},-}(t')) - \int_0^t {\rm d}t' \left\langle {\rm F}, A(t') \left| \frac{\partial}{\partial t'} \right| {\rm F}, A(t') \right\rangle \right\}.$$

For $t = T$, $|{\rm F}, A(T)\rangle = |{\rm F}, A(0)\rangle$ (the instantaneous eigenfunctions are chosen to be periodic in time) and

$$\psi_{\rm F}(T) = \exp\{{\rm i}\gamma_{\rm F}^{\rm dyn} + {\rm i}\gamma_{\rm F}^{\rm Berry}\}\psi_{\rm F}(0)$$

where

$$\gamma_{\rm F}^{\rm dyn} \equiv -\frac{1}{\hbar} \int_0^T {\rm d}t \, (\varepsilon_{{\rm F},+}(t) + \varepsilon_{{\rm F},-}(t))$$

while

$$\gamma_{\rm F}^{\rm Berry} = \gamma_{{\rm F},+}^{\rm Berry} + \gamma_{{\rm F},-}^{\rm Berry}$$
$$\gamma_{{\rm F},\pm}^{\rm Berry} \equiv \int_0^T {\rm d}t \int_{-{\rm L}/2}^{{\rm L}/2} {\rm d}x \, \dot{A}_1(x,t) \left\langle {\rm F}, A(t); \pm \left| {\rm i}\frac{\delta}{\delta A_1(x,t)} \right| {\rm F}, A(t); \pm \right\rangle \tag{13}$$

is Berry's phase [13].

If we define the U(1) connections

$$\mathcal{A}_{{\rm F},\pm}(x,t) \equiv \left\langle {\rm F}, A(t); \pm \left| {\rm i}\frac{\delta}{\delta A_1(x,t)} \right| {\rm F}, A(t); \pm \right\rangle \tag{14}$$

then

$$\gamma_{{\rm F},\pm}^{\rm Berry} = \int_0^T {\rm d}t \int_{-{\rm L}/2}^{{\rm L}/2} {\rm d}x \, \dot{A}_1(x,t)\mathcal{A}_{{\rm F},\pm}(x,t).$$

We see that upon parallel transport around a closed loop on $\mathcal{A}^1$ the Fock states $|{\rm F}, A(t); \pm\rangle$ acquire an additional phase which is integrated exponential of $\mathcal{A}_{{\rm F},\pm}(x,t)$. Whereas the dynamical phase $\gamma_{\rm F}^{\rm dyn}$ provides information about the duration of the evolution, the Berry's phase reflects the nontrivial holonomy of the Fock states on $\mathcal{A}^1$.

However, a direct computation of the diagonal matrix elements of $\frac{\delta}{\delta A_1(x,t)}$ in (13) requires a globally single-valued basis for the eigenstates $|{\rm F}, A(t); \pm\rangle$ which is not available. The connections in (14) can be defined only locally on $\mathcal{A}^1$, in regions where $[\frac{e_+ b{\rm L}}{2\pi\hbar}]$ is fixed. The values of A_1 in regions of different $[\frac{e_+ b{\rm L}}{2\pi\hbar}]$ are connected by topologically nontrivial gauge transformations. If $[\frac{e_+ b{\rm L}}{2\pi\hbar}]$ changes, then there is a nontrivial spectral flow, i.e. some energy levels of the first quantized fermionic Hamiltonians cross zero and change sign. This means that the definition of the Fock vacuum of the second quantized fermionic Hamiltonian changes (see equations (8) and (9) and figure 3). Since the creation and annihilation operators $a^\dagger, a$ (and $b^\dagger, b$) are continuous functionals of $A_1(x)$, the definition of all excited Fock states $|{\rm F}, A(t)\rangle$ is also discontinuous. The connections $\mathcal{A}_{{\rm F},\pm}$ are not therefore well defined globally. Their global characterization necessiates the usual introduction of transition functions.

Furthermore, $\mathcal{A}_{{\rm F},\pm}$ are not invariant under A-dependent redefinitions of the phases of the Fock states: $|{\rm F}, A(t); \pm\rangle \to \exp\{-{\rm i}\chi_\pm[A]\}|{\rm F}, A(t); \pm\rangle$, and transform like a U(1) vector potential

$$\mathcal{A}_{{\rm F},\pm} \to \mathcal{A}_{{\rm F},\pm} + \frac{\delta\chi_\pm[A]}{\delta A_1}.$$

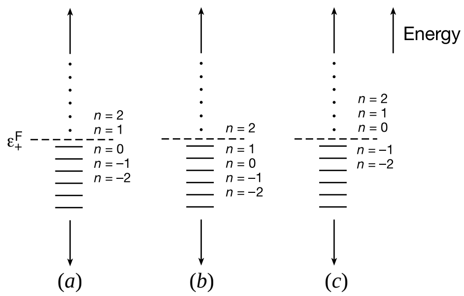

Figure 3. Schematic representation of the vacuum state at different values of $\left[\frac{e_+bL}{2\pi\hbar}\right]$: (*a*) $\left[\frac{e_+bL}{2\pi\hbar}\right]=0$, (*b*) $\left[\frac{e_+bL}{2\pi\hbar}\right]=1$, (*c*) $\left[\frac{e_+bL}{2\pi\hbar}\right]=-1$. Only the positive chirality sector is shown.

For these reasons, to calculate $\gamma_{\rm F}^{\rm Berry}$ it is more convenient to compute first the U(1) curvature tensors

$$\mathcal{F}_{\rm F}^{\pm}(x,y,t) \equiv \frac{\delta}{\delta A_1(x,t)}\mathcal{A}_{{\rm F},\pm}(y,t) - \frac{\delta}{\delta A_1(y,t)}\mathcal{A}_{{\rm F},\pm}(x,t) \tag{15}$$

and then deduce $\mathcal{A}_{{\rm F},\pm}$.

3.1. *n-particle states* ($n \geqslant 3$)

For n-particle states $|m_1; m_2; \ldots; m_n; A; \pm\rangle$ $(m_1 < m_2 < \cdots < m_n)$, the U(1) curvature tensors are

$$\begin{aligned}\mathcal{F}^{\pm}_{m_1,m_2,\ldots,m_n}(x,y,t) = {\rm i}\sum_{k=0}^{\infty}\frac{1}{k!}&\sum_{\overline{m}_1,\overline{m}_2,\ldots,\overline{m}_k\in\mathcal{Z}}\\ &\times\left\{\left\langle m_1; m_2 : \ldots; m_n; A; \pm\left|\frac{\delta}{\delta A_1(x,t)}\right|\overline{m}_1; \overline{m}_2; \ldots; \overline{m}_k; A; \pm\right\rangle\right.\\ &\times\left.\left\langle \overline{m}_1; \overline{m}_2; \ldots; \overline{m}_k; A; \pm\left|\frac{\delta}{\delta A_1(y,t)}\right|m_1; m_2; \ldots; m_n; A; \pm\right\rangle - (x\leftrightarrow y)\right\}\end{aligned}$$

where the completeness condition (12) is inserted.

Since

$$\begin{aligned}&\left\langle m_1; m_2; \ldots; m_n; A; \pm\left|\frac{\delta : \hat{\rm H}_\pm :}{\delta A_1(x,t)}\right|\overline{m}_1; \overline{m}_2; \ldots; \overline{m}_k; A; \pm\right\rangle\\ &\qquad = \left\{\sum_{i=1}^{k}\varepsilon_{\overline{m}_i,\pm}{\rm sign}(\varepsilon_{\overline{m}_i,\pm}) - \sum_{i=1}^{n}\varepsilon_{m_i,\pm}{\rm sign}(\varepsilon_{m_i,\pm})\right\}\\ &\qquad\quad\times\left\langle m_1; m_2; \ldots; m_n; A; \pm\left|\frac{\delta}{\delta A_1(x,t)}\right|\overline{m}_1; \overline{m}_2; \ldots; \overline{m}_k; A; \pm\right\rangle\end{aligned}$$

and $:\hat{\rm H}_\pm:$ are quadratic in the positive and negative chirality creation and annihilation operators, the matrix elements $\langle m_1; m_2; \ldots; m_n; A; \pm|\frac{\delta}{\delta A_1(x,t)}|\overline{m}_1; \overline{m}_2; \ldots; \overline{m}_k; A; \pm\rangle$ and so the corresponding curvature tensors $\mathcal{F}^{\pm}_{m_1,m_2,\ldots,m_n}$ and Berry phases $\gamma^{\rm Berry}_{m_1,m_2,\ldots,m_n;\pm}$ vanish for all values of $m_i (i=\overline{1,n})$ for $n \geqslant 3$.

3.2. 2-particle states

For 2-particle states $|m_1; m_2; A; \pm\rangle$ $(m_1 < m_2)$, only the vacuum state survives in the completeness condition inserted so that the curvature tensors $\mathcal{F}^{\pm}_{m_1 m_2}$ take the form

$$\mathcal{F}^{\pm}_{m_1 m_2}(x,y,t) = \frac{\mathrm{i}}{\hbar^2} \frac{1}{(\varepsilon_{m_1,\pm}\mathrm{sign}(\varepsilon_{m_1,\pm}) + \varepsilon_{m_2,\pm}\mathrm{sign}(\varepsilon_{m_2,\pm}))^2} \times \left\{ \left\langle m_1; m_2; A; \pm \left| \frac{\delta : \hat{\mathrm{H}}_{\pm} :}{\delta A_1(y,t)} \right| \mathrm{vac}; A; \pm \right\rangle \times \left\langle \mathrm{vac}; A; \pm \left| \frac{\delta : \hat{\mathrm{H}}_{\pm} :}{\delta A_1(x,t)} \right| m_1; m_2; A; \pm \right\rangle - (x \leftrightarrow y) \right\}.$$

With $: \hat{\mathrm{H}}_{\pm} : (t)$ given by (10), $\mathcal{F}^{\pm}_{m_1 m_2}$ are evaluated as

$$\mathcal{F}^{\pm}_{m_1 m_2} = \begin{cases} 0 & \text{for } m_1, m_2 > \left[\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}\right] \\ & \text{and } m_1, m_2 \leqslant \left[\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}\right] \\ \mp \frac{e_{\pm}^2}{2\pi^2\hbar^2} \frac{1}{(m_2 - m_1)^2} \times \sin\left\{\frac{2\pi}{\mathrm{L}}(m_2 - m_1)(x - y)\right\} & \text{for } m_1 \leqslant \left[\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}\right], m_2 > \left[\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}\right] \end{cases}$$

i.e. the curvatures are nonvanishing only for states with one particle and one hole.

The corresponding connections are easily deduced as

$$\mathcal{A}^{\pm}_{m_1 m_2}(x,t) = -\tfrac{1}{2} \int_{-\mathrm{L}/2}^{\mathrm{L}/2} \mathrm{d}y\, \mathcal{F}^{\pm}_{m_1 m_2}(x,y,t) A_1(y,t).$$

The Berry phases become

$$\gamma^{\mathrm{Berry}}_{m_1 m_2,\pm} = -\tfrac{1}{2} \int_0^{\mathrm{T}} \mathrm{d}t \int_{-\mathrm{L}/2}^{\mathrm{L}/2} \mathrm{d}x \int_{-\mathrm{L}/2}^{\mathrm{L}/2} \mathrm{d}y\, \dot{A}_1(x,t) \mathcal{F}^{\pm}_{m_1 m_2}(x,y,t) A_1(y,t).$$

If we introduce the Fourier expansion for the gauge field

$$A_1(x,t) = b(t) + \sum_{\substack{p \in \mathcal{Z} \\ p \neq 0}} \mathrm{e}^{\mathrm{i}\frac{2\pi}{\mathrm{L}} p x} \alpha_p(t)$$

then in terms of the gauge field Fourier components the Berry phases take the form

$$\gamma^{\mathrm{Berry}}_{m_1 m_2,\pm} = \mp \frac{e_{\pm}^2 \mathrm{L}^2}{8\pi^2\hbar^2} \frac{1}{(m_2 - m_1)^2} \int_0^{\mathrm{T}} \mathrm{d}t\, \mathrm{i}(\alpha_{m_2 - m_1}\dot{\alpha}_{m_1 - m_2} - \alpha_{m_1 - m_2}\dot{\alpha}_{m_2 - m_1})$$

for $m_1 \leqslant [\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}], m_2 > [\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}]$, vanishing for $m_1, m_2 > [\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}]$ and $m_1, m_2 \leqslant [\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}]$. Therefore, a parallel transportation of the states $|m_1; m_2; A; \pm\rangle$ with two particles or two holes around a closed loop in (α_p, α_{-p})-space $(p > 0)$ yields back the same states, while the states with one particle and one hole are multiplied by the phases $\gamma^{\mathrm{Berry}}_{m_1 m_2,\pm}$.

For the Schwinger model when $N = 1$ and $e_+ = e_-$ as well as for axial electrodynamics when $N = -1$ and $e_+ = -e_-$, the nonvanishing Berry phases for the positive and negative chirality 2-particle states are opposite in sign,

$$\gamma^{\mathrm{Berry}}_{m_1 m_2,+} = -\gamma^{\mathrm{Berry}}_{m_1 m_2,-}$$

so that for the states $|m_1; m_2; A\rangle = |m_1; m_2; A; +\rangle \otimes |m_1; m_2; A; -\rangle$ the total Berry phase is zero.

3.3. 1-particle states

For 1-particle states $|m; A; \pm\rangle$, the U(1) curvature tensors are

$$\mathcal{F}_m^{\pm}(x, y, t) = \mathrm{i} \sum_{\substack{\overline{m} \in \mathcal{Z} \\ \overline{m} \neq m}} \frac{1}{\hbar^2} \frac{1}{(\varepsilon_{\overline{m},\pm}\mathrm{sign}(\varepsilon_{\overline{m},\pm}) - \varepsilon_{m,\pm}\mathrm{sign}(\varepsilon_{m,\pm}))^2}$$
$$\times \left\{ \left\langle m; A; \pm \left| \frac{\delta : \hat{\mathrm{H}}_{\pm} :}{\delta A_1(y, t)} \right| \overline{m}; A; \pm \right\rangle \right.$$
$$\left. \times \left\langle \overline{m}; A; \pm \left| \frac{\delta : \hat{\mathrm{H}}_{\pm} :}{\delta A_1(x, t)} \right| m; A; \pm \right\rangle - (x \longleftrightarrow y) \right\}.$$

By a direct calculation we easily obtain

$$\mathcal{F}^{\pm}_{m > [\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}]} = \sum_{\overline{m} = m - [\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}]}^{\infty} \mathcal{F}^{\pm}_{0\overline{m}}$$
$$\mathcal{F}^{\pm}_{m \leqslant [\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}]} = \sum_{\overline{m} = [\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}] - m + 1}^{\infty} \mathcal{F}^{\pm}_{0\overline{m}}$$

where $\mathcal{F}^{\pm}_{0\overline{m}}$ are curvature tensors for the 2-particle states $|0; \overline{m}; A; \pm\rangle$ $(\overline{m} > 0)$.

The Berry phases acquired by the states $|m; A; \pm\rangle$ by their parallel transportation around a closed loop in (α_p, α_{-p})-space $(p > 0)$ are

$$\gamma_{\pm}^{\mathrm{Berry}} \left(m > \left[\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar} \right] \right) = \sum_{\overline{m} = m - [\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}]}^{\infty} \gamma_{0\overline{m};\pm}^{\mathrm{Berry}}$$
$$\gamma_{\pm}^{\mathrm{Berry}} \left(m \leqslant \left[\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar} \right] \right) = \sum_{\overline{m} = [\frac{e_{\pm} b \mathrm{L}}{2\pi\hbar}] - m + 1}^{\infty} \gamma_{0\overline{m};\pm}^{\mathrm{Berry}}$$

where $\gamma_{0\overline{m};\pm}^{\mathrm{Berry}}$ are phases acquired by the states $|0; \overline{m}; A; \pm\rangle$ by the same transportation.

For the $N = \pm 1$ models, the total 1-particle curvature tensor $\mathcal{F}_m = \mathcal{F}_m^{+} + \mathcal{F}_m^{-}$ and total Berry phase $\gamma^{\mathrm{Berry}} = \gamma_{+}^{\mathrm{Berry}} + \gamma_{-}^{\mathrm{Berry}}$ vanish.

3.4. vacuum states

For the vacuum case, only 2-particle states contribute to the sum of the completeness condition, so the vacuum curvature tensors are

$$\mathcal{F}^{\pm}_{\mathrm{vac}}(x, y, t) = -\tfrac{1}{2} \sum_{\overline{m}_1; \overline{m}_2 \in \mathcal{Z}} \mathcal{F}^{\pm}_{\overline{m}_1 \overline{m}_2}(x, y, t).$$

Taking the sums, we obtain

$$\mathcal{F}^{\pm}_{\mathrm{vac}} = \pm \frac{e_{+}^2}{2\pi\hbar^2} \sum_{n > 0} \left(\frac{1}{2} \epsilon(x - y) - \frac{1}{\mathrm{L}}(x - y) \right). \tag{16}$$

The total vacuum curvature tensor

$$\mathcal{F}_{\mathrm{vac}} = \mathcal{F}^{+}_{\mathrm{vac}} + \mathcal{F}^{-}_{\mathrm{vac}} = (1 - N^2) \frac{e_{+}^2}{2\pi\hbar^2} \left(\frac{1}{2} \epsilon(x - y) - \frac{1}{\mathrm{L}}(x - y) \right)$$

vanishes for $N = \pm 1$.

The corresponding U(1) connection is deduced as

$$\mathcal{A}_{\text{vac}}(x,t) = -\frac{1}{2}\int_{-\text{L}/2}^{\text{L}/2} \mathrm{d}y\, \mathcal{F}_{\text{vac}}(x,y,t) A_1(y,t)$$

so the total vacuum Berry phase is

$$\gamma_{\text{vac}}^{\text{Berry}} = -\tfrac{1}{2}\int_0^T \mathrm{d}t \int_{-\text{L}/2}^{\text{L}/2} \mathrm{d}x \int_{-\text{L}/2}^{\text{L}/2} \mathrm{d}y\, \dot{A}_1(x,t)\mathcal{F}_{\text{vac}}(x,y,t)A_1(y,t)$$

For $\text{N} = 0$ and in the limit $\text{L} \to \infty$, when the second term in (16) may be neglected, the U(1) curvature tensor coincides with that obtained in [5, 15], while the Berry phase becomes

$$\gamma_{\text{vac}}^{\text{Berry}} = \frac{1}{\hbar}\int_0^T \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x\, \mathcal{L}_{\text{nonlocal}}(x,t)$$

where

$$\mathcal{L}_{\text{nonlocal}}(x,t) \equiv -\frac{e_+^2}{8\pi^2\hbar}\int_{-\infty}^{\infty} \mathrm{d}y\, \dot{A}_1(x,t)\epsilon(x-y)A_1(y,t)$$

is a nonlocal part of the effective Lagrange density of the CSM [16]. The effective Lagrange density is a sum of the ordinary Lagrange density of the CSM and the nonlocal part $\mathcal{L}_{\text{nonlocal}}$. As shown in [16], the effective Lagrange density is equivalent to the ordinary one in the sense that the corresponding preliminary Hamiltonians coincide on the constrained submanifold $\text{G} \approx 0$. This equivalence is valid at the quantum level, too. If we start from the effective Lagrange density and apply appropriately the Dirac quantization procedure, then we come to a quantum theory which is exactly the quantum theory obtained from the ordinary Lagrange density. We therefore obtain that the Berry phase is an action and that the CSM can be defined equivalently by both the effective action with the Berry phase included and the ordinary one without the Berry phase.

In terms of the gauge field Fourier components, the connection $\mathcal{A}_{\text{vac}}$ is rewritten as

$$\left\langle \text{vac};\, A(t) \left| \frac{\mathrm{d}}{\mathrm{d}b(t)} \right| \text{vac};\, A(t) \right\rangle = 0$$

$$\left\langle \text{vac};\, A(t) \left| \frac{\mathrm{d}}{\mathrm{d}\alpha_{\pm p}(t)} \right| \text{vac};\, A(t) \right\rangle \equiv \mathcal{A}_{\text{vac};\pm}(p,t) = \pm(1-N^2)\frac{e_+^2\text{L}^2}{8\pi^2\hbar^2}\frac{1}{p}\alpha_{\mp p}$$

so the nonvanishing vacuum curvature is

$$\mathcal{F}_{\text{vac}}(p) \equiv \frac{\mathrm{d}}{\mathrm{d}\alpha_{-p}}\mathcal{A}_{\text{vac};+} - \frac{\mathrm{d}}{\mathrm{d}\alpha_p}\mathcal{A}_{\text{vac};-} = (1-N^2)\frac{e_+^2\text{L}^2}{4\pi^2\hbar^2}\frac{1}{p}.$$

The total vacuum Berry phase becomes

$$\gamma_{\text{vac}}^{\text{Berry}} = \int_0^{\text{T}} \mathrm{d}t \sum_{p>0} \mathcal{F}_{\text{vac}}(p) i\alpha_p\dot{\alpha}_{-p}.$$

For the $N \neq \pm 1$ models where the local gauge symmetry is known to be realized projectively [4], the vacuum Berry phase is nonzero. For $N = \pm 1$ when the representation is unitary, the curvature $\mathcal{F}_{\text{vac}}(p)$ and the vacuum Berry phase vanish.

The projective representation of the local gauge symmetry is responsible for anomaly. In the full quantized theory of the CSM when the gauge fields are also quantized the physical states respond to gauge transformations from the zero topological class with a phase [4]. This phase contributes to the commutator of the Gauss law generators by a Schwinger term and produces therefore an anomaly.

A connection of the nonvanishing vacuum Berry phase to the projective representation can be shown in a more direct way. Under the topologically trivial gauge transformations, the gauge field Fourier components α_p, α_{-p} transform as follows

$$\alpha_p \stackrel{\tau}{\to} \alpha_p - \mathrm{i}p\tau_-(p)$$
$$\alpha_{-p} \stackrel{\tau}{\to} \alpha_{-p} - \mathrm{i}p\tau_+(p)$$

where $\tau_\pm(p)$ are smooth gauge parameters.

The nonlocal Lagrangian

$$\mathrm{L}_{\mathrm{nonlocal}}(t) \equiv \int_{-\mathrm{L}/2}^{\mathrm{L}/2} \mathrm{d}x\, \mathcal{L}_{\mathrm{nonlocal}}(x,t) = \hbar \sum_{p>0} \mathcal{F}_{\mathrm{vac}}(p) \mathrm{i}\alpha_p \dot{\alpha}_{-p}$$

changes as

$$\mathrm{L}_{\mathrm{nonlocal}}(t) \stackrel{\tau}{\to} \mathrm{L}_{\mathrm{nonlocal}}(t) - 2\pi\hbar \frac{\mathrm{d}}{\mathrm{d}t}\alpha_1(A;\tau)$$

where

$$\alpha_1(A;\tau) \equiv -\frac{1}{4\pi} \sum_{p>0} p\mathcal{F}_{\mathrm{vac}}(p)(\alpha_{-p}\tau_- - \alpha_p\tau_+)$$

is just 1-cocycle occurring in the projective representation of the gauge group. This examplifies a connection between the nonvanishing vacuum Berry phase and the fact that the local gauge symmetry is realized projectively.

4. Conclusions

Let us summarize.

(i) We have calculated explicitly the Berry phase and the corresponding U(1) connection and curvature for the fermionic vacuum and many particle Fock states. For the $N \neq \pm 1$ models, we obtain that the Berry phase is nonzero for the vacuum, 1- and 2-particle states with one particle and one hole. For all other many particle states the Berry phase vanishes. This is caused by the form of the second quantized fermionic Hamiltonian which is quadratic in the positive and negative chirality creation and annihilation operators.

(ii) For the $N = \pm 1$ models without anomaly, i.e. for the SM and axial electrodynamics, the Berry phases acquired by the negative and positive chirality parts of the Fock states are opposite in sign and cancel each other, so that the total Berry phase for all Fock states is zero.

(iii) A connection between the Berry phase and anomaly becomes more explicit for the vacuum state. We have shown that for our model the vacuum Berry phase contributes to the effective action, being that additional part of the effective action which differs it from the ordinary one. Under the topologically trivial gauge transformations the corresponding addition in the effective Lagrangian changes by a total time derivative of the gauge group 1-cocycle occurring in the projective representation. This demonstrates an interrelation between the Berry phase, anomaly and effective action.

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References

[1] Jackiw R and Rajaraman R 1985 *Phys. Rev. Lett.* **54** 1219
Rajaraman R 1985 *Phys. Lett.* **154B** 305
[2] Faddeev L D and Shatashvili S L 1986 *Phys. Lett.* **167B** 225
Faddeev L D 1984 *Phys. Lett.* **14B** 81 (1984)
[3] Niemi A and Semenoff G 1985 *Phys. Rev. Lett.* **55** 927
Niemi A and Semenoff G 1986 *Phys. Lett.* **56** 1019
[4] Saradzhev F M 1991 *Int. J. Mod. Phys.* A **6** 3823
Saradzhev F M 1994 *Int. J. Mod. Phys.* A **9** 3179
Saradzhev F M 1992 *Phys. Lett.* B **278** 449
Saradzhev F M 1994 *Phys. Lett.* B **324** 192
[5] Niemi A and Semenoff G 1986 *Phys. Lett.* **175B** 439
[6] Halliday I G, Rabinovici E and Schwimmer A 1986 *Nucl. Phys.* B **268** 413
[7] Saradzhev F M 1996 *Phys. Lett.* B **372** 283
[8] Nelson P and Alvarez-Gaume L 1985 *Commun. Math. Phys.* **99** 103
[9] Niemi A and Semenoff G 1986 *Phys. Rep.* **135** 99
[10] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University)
[11] Feynman R 1972 *Statistical Mechanics—A Set of Lectures* (MA: Benjamin)
[12] Schiff L I 1968 *Quant. Mech.* (New York: McGraw-Hill)
[13] Berry M V 1984 *Proc. R. Soc.* A **392** 45
[14] Zwanziger J W *et al* 1990 *Annu. Rev. Phys. Chem.* **41** 601
[15] Semenoff G 1987 *Super Field Theory* ed H Lee *et al* (New York: Plenum)
[16] Saradzhev F M 1993 *Int. J. Mod. Phys.* A **8** 2915, 2937